• Laplace equation in polar form

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

2- The complex logarithm function.

$$e^{w} = z \Rightarrow \quad w = \ln z$$

$$\ln z = \ln r + i(\theta + 2n\pi)$$ (multi-valued fun.)

$n = 0, \pm 1, \pm 2, \dots$

• Principle value of $\ln z$

$n = 0$ $\quad \operatorname{Ln} z = \ln r + i\theta \quad , \; -\pi < \theta \leq \pi$

• properities

1- $\ln z_1 z_2 = \ln z_1 + \ln z_2$

2- $\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$

3- $\frac{d(\ln z)}{dz} = \frac{1}{z}$

3- Complex power funs.

$$z^{\alpha} = e^{\ln z^{\alpha}} = e^{\alpha \ln z}$$

نفس خواص دالـة الـ ln

$\xrightarrow{\text{تفاضلها}} \alpha z^{\alpha - 1}$

$$\alpha^{z} = e^{\ln \alpha^{z}} = e^{z \ln \alpha}$$

نفس خواص ال expon.

$\xrightarrow{\text{تفاضلها}} \alpha^{z} \ln \alpha$

*where : α is a complex constant.

1- $z^{\alpha} z = z^{\alpha + \beta}$

2- $\frac{z^{\alpha}}{z^{\beta}} = z^{\alpha - \beta}$

3- $(z^{\alpha})^{\beta} = z^{\alpha\beta}$

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1- Trigonometric and hyperbolic funs. الدوال المثلثية والزائدية

- $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$
- $\sin(-z) = \sin(z)$ ↳ odd fun.
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$
- $\sin 2z = 2 \sin z \cos z$
- $\sin(z + \frac{\pi}{2}) = \cos(z)$
- $\sin(z - \frac{\pi}{2}) = -\cos(z)$
- $\sin(z + 2\pi) = \sin z$
- $\sin(z + \pi) = -\sin z$

$\sin z = \sin x \cosh y + i \cos x \sinh y$

- $\cos^2 z + \sin^2 z = 1$
- $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$\sin z = 0 \Rightarrow z = n\pi$

- $\dfrac{d}{dz}(\sin z) = \cos z$

- $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$
- $\cos(-z) = \cos(z)$ ↳ even fun.
- $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin$
- $\cos 2z = \cos^2 z - \sin^2 z$
- $\cos(z + 2\pi) = \cos z$
- $\cos(z + \pi) = -\cos z$
- $\cos z = \cos x \cosh y - i \sin x \sinh y$

$\sin(iy) = i \sinh(y)$

$\cos(iy) = \cosh(y)$

- $|\cos z|^2 = \cos^2 x + \sinh^2 y$
- $\cos z = 0 \Rightarrow z = \frac{\pi}{2} + n\pi$

$\dfrac{d}{dz}(\cos z) = -\sin(z)$

$\sec z = \dfrac{1}{\cos z} \xrightarrow{\text{تفاضل}} \sec z \tan z$

$\operatorname{cosec} z = \dfrac{1}{\sin z} \xrightarrow{\text{تفاضل}} -\operatorname{cosec} z \cot z$

$\cot z = \dfrac{1}{\tan z} \xrightarrow{\text{تفاضل}} -\operatorname{cosec}^2 z$

$\tan z = \dfrac{\sin z}{\cos z} \xrightarrow{\text{تفاضل}} \sec^2 z$

Period π

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$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

- $\sinh(-z) = -\sinh z$
  odd fun.

- $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

$$\sinh(iz) = i\sin(z)$$

- $\frac{d}{dz}(\sinh z) = \cosh(z)$

- $\sinh z = \sinh x \cos y + i \cosh x \sin y$

- $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

- $\sinh z = 0 \Rightarrow z = n\pi i$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

- $\cosh(-z) = \cosh(z)$
  even fun.

- $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

$$\cosh(iz) = \cos(z)$$

- $\frac{d}{dz}(\cosh(z)) = \sinh(z)$

- $\cosh z = \cosh x \cos y + i \sinh x \sin y$

- $|\cosh z|^2 = \sinh^2 x + \cos^2 y$

- $\cosh z = 0 \Rightarrow z = (\frac{\pi}{2} + n\pi)i$

- $\cosh^2 z - \sinh^2 z = 1$

- $\tanh z = \frac{\sinh z}{\cosh z} \xrightarrow{\text{تفاضلها}} \operatorname{sech}^2 z$
- $\coth z \xrightarrow{\text{تفاضلها}} -\operatorname{cosech}^2 z$
- $\operatorname{sech} z \xrightarrow{\text{تفاضلها}} -\operatorname{sech} z \tanh z$
- $\operatorname{cosech} z \xrightarrow{\text{تفاضلها}} -\operatorname{cosech} z \coth z$

5- Inverse Functions.

1- $\sin^{-1} z = -i \ln(iz + \sqrt{1 - z^2})$

2- $\cos^{-1} z = -i \ln(z + i\sqrt{1 - z^2})$

3- $\tan^{-1} z = \frac{i}{2} \ln \frac{i + z}{i - z}$

4- $\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1})$

5- $\cosh^{-1} z = \ln(z + \sqrt{z^2 - 1})$

6- $\tanh^{-1} z = \frac{1}{2} \ln \frac{1 + z}{1 - z}$

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- $\frac{d}{dz}\sin^{-1}z = \frac{1}{\sqrt{1-z^2}}$
- $\frac{d}{dz}\cos^{-1}z = \frac{-1}{\sqrt{1-z^2}}$
- $\frac{d}{dz}\tan^{-1}z = \frac{1}{1+z^2}$

- $\frac{d}{dz}\sinh^{-1}z = \frac{1}{\sqrt{z^2+1}}$
- $\frac{d}{dz}\cosh^{-1}z = \frac{1}{\sqrt{z^2-1}}$
- $\frac{d}{dz}\tanh^{-1}z = \frac{1}{1-z^2}$

- Express in terms of $u+iv$ $\ell n z$ if:

1- $z = (1+i)^4$

Sol.

$$z = (\sqrt{2}\, e^{\frac{i\pi}{4}})^4 = 4e^{i\pi} \Rightarrow r = 4 \;\&\; \theta = \pi$$

$$\therefore \ell n\, z = \ell n\, r + i(\theta + 2n\pi)$$

$$\ell n z = \ell n 4 + i(\pi + 2n\pi)$$

2- Solve for "z" the following eqn.

$\ell n z = 2 + \frac{\pi}{4} i$

Sol.

$$\ell n r + i(\theta + 2n\pi) = 2 + \frac{\pi}{4} i$$

$$\therefore \ell n r = 2 \Rightarrow r = e^2$$

$$\theta + 2n\pi = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4} + 2n\pi$$

3- Show that $\ell n\left(\frac{x+iy}{x-iy}\right) = 2i\tan^{-1}\left(\frac{y}{x}\right)$

Sol.

$$\ell n\left(\frac{x+iy}{x-iy}\right) = \ell n\left(\frac{re^{i\theta}}{re^{-i\theta}}\right) = \ell n\, e^{2i\theta} = 2i\theta$$

$$= 2i\tan^{-1}\frac{y}{x} \;\#$$

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4- Complete the principle value for $\ln(z)$ if

1- $Z = (1+i)$

Sol.

$r = \sqrt{2}$, $\theta = \tan^{-1} 1 = \frac{\pi}{4}$

$\therefore \boxed{\ln Z = \ln\sqrt{2} + i\frac{\pi}{4}}$

2- $Z = -3-4i$

Sol.

$r = \sqrt{25} = 5$ & $\theta = \tan^{-1}\frac{-4}{-3} = 1.704\pi$

$\boxed{\ln Z = \ln 5 - i0.295\pi}$

5- Solve the following eqn.

$Z = (1+i)^i$

Sol.

$\ln Z = i\ln(1+i)$

$= i\left((\ln\sqrt{2}) + i(\frac{\pi}{4} + 2n\pi)\right)$

$\ln r + i(\theta + 2n\pi) = i\ln\sqrt{2} - (\frac{\pi}{4} + 2n\pi)$

$\therefore r = e^{-(\frac{\pi}{4}+2n\pi)}$

$\theta = \ln\sqrt{2} + 2n\pi$

6- Express in terms of $u+iv$

- $\cos(2-4i)$

$= \cos 2\cos 4i + \sin 2\sin 4i$

$= \boxed{\cos 2\cosh 4 + i\sin 2\sinh 4}$

- $\sinh(-2+3i)$

$= \sinh(-2)\cosh(3i) + \cosh(-2)\sinh(3i)$

$= -\sinh(2)\cos 3 + i\cosh(2)\sin(3)$

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7- Solve the Following eqns.

- $\cosh z = \frac{1}{2}$

Sol.

$$\frac{e^{z}+e^{-z}}{2}=\frac{1}{2} \Rightarrow e^{z}+e^{-z}=1 \qquad * e^{z},$$

$$(e^{z})^{2}-e^{z}+1=0$$

$$\therefore e^{z}=\frac{1\pm\sqrt{1-4}}{2}=\frac{1}{2}\pm\frac{\sqrt{3}i}{2}$$

$e^{z}=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$r=1$ & $\theta=\frac{\pi}{3}$

$e^{z}=e^{i\frac{\pi}{3}}$

$e^{z+2n\pi i}=e^{i\frac{\pi}{3}}$

$\therefore z+2n\pi i=i\frac{\pi}{3}$

$x=0$ & $y=\frac{\pi}{3}+2n\pi$

$e^{z}=\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$r=1$ & $\theta=\frac{5\pi}{3}$

$e^{z}=e^{i\frac{5\pi}{3}}$

$e^{z+2n\pi i}=e^{i\frac{5\pi}{3}}$

$z+2n\pi i=i\frac{5\pi}{3}$

$\therefore x=0$ & $y=\frac{5\pi}{3}+2n\pi$

- $\sin z = 10$

Sol.

$$\frac{e^{iz}-e^{-iz}}{2i}=10 \Rightarrow e^{iz}-e^{-iz}=20i \qquad * e^{iz}$$

$$(e^{iz})^{2}-20ie^{iz}-1=0$$

$$\therefore e^{iz}=\frac{20i\pm\sqrt{400+4}}{2}=\frac{20i\pm\sqrt{396}i}{2}=\frac{20i\pm19.89i}{2}$$

$e^{iz}=19.94i$ & $e^{iz}=0.05i$

$r=19.94$ & $\theta=\frac{\pi}{2}$ & $r=0.05$ & $\theta=\frac{\pi}{2}$

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$$e^{iz} = 19.94\, e^{i\pi/2}$$

$$e^{iz + 2n\pi i} = 19.94\, e^{i\frac{\pi}{2}}$$

$$\therefore \ln 19.94 + i\frac{\pi}{2} = iz + 2n\pi i$$

$$\therefore (x + 2n\pi)i - y = \ln 19.94 + i\frac{\pi}{2}$$

$$\therefore x = \frac{\pi}{2} + 2n\pi \quad \& \quad y = -\ln 19.94$$

$$e^{iz} = 0.05\, e^{i\pi/2}$$

$$e^{iz + 2n\pi i} = 0.05\, e^{i\frac{\pi}{2}}$$

$$iz + 2n\pi i = \ln 0.05 + i\frac{\pi}{2}$$

$$(x + 2n\pi i)i - y = \ln 0.05 + i\frac{\pi}{2}$$

$$\therefore x = \frac{\pi}{2} + 2n\pi \quad \& \quad y = -\ln 0.05$$

8- Find the value of $(\cosh(1+i))^{5+6i}$

Sol.

$$\cosh(1+i) = \cosh 1 \cosh i + \sinh 1 \sinh i$$

$$= \cosh 1 \cos 1 + i \sinh 1 \sin 1$$

$$= 0.83322 + i\, 0.9888$$

$$\therefore (\cosh(1+i))^{5+6i} = (0.83322 + i\, 0.9888)^{5+6i}$$

$$= e^{5+6i \ln(0.83322 + i\, 0.9888)}$$

$$\ln(0.83322 + i\, 0.9888) = \ln r + i(\theta + 2n\pi)$$

تكملة سابق

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• Show that $\sin^{-1} z = -i \ln(iz + \sqrt{1-z^2})$

Sol.

$\sin^{-1} z = \omega \Rightarrow z = \sin\omega$

$\therefore \dfrac{e^{i\omega} - e^{-i\omega}}{2i} = z \Rightarrow e^{i\omega} - e^{-i\omega} = 2iz \quad (* e^{i\omega})$

$(e^{i\omega})^2 - 2iz\, e^{i\omega} - 1 = 0$

$\therefore e^{i\omega} = \dfrac{\cancel{2}iz \pm \sqrt{-4z^2+4}}{\cancel{2}}$

$e^{i\omega} = iz \pm \sqrt{1-z^2} \Rightarrow i\omega = \ln(iz + \sqrt{1-z^2})$

refused

$\therefore \sin^{-1} z = -i \ln(iz + \sqrt{1-z^2})$

• Show that $\cos^{-1} z = -i \ln(z + i\sqrt{1-z^2})$

Sol.

$\cos^{-1} z = \omega \Rightarrow \cos\omega = z$

$\therefore \dfrac{e^{i\omega} + e^{-i\omega}}{2} = z \Rightarrow e^{i\omega} + e^{-i\omega} = 2z \quad (* e^{i\omega})$

$\therefore (e^{i\omega})^2 - 2z\, e^{i\omega} + 1 = 0$

$e^{i\omega} = \dfrac{2z \pm \sqrt{4z^2-4}}{2} = (z \pm i\sqrt{1-z^2})$ refused

$\therefore i\omega = \ln(z + i\sqrt{1-z^2})$

$\therefore \cos^{-1} z = -i \ln(z + i\sqrt{1-z^2})$

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• Show that $\tan^{-1} z = \frac{1}{2} \ln\left(\frac{i+z}{i-z}\right)$

Sol.

$$\tan^{-1} z = \omega \Rightarrow z = \tan\omega = \frac{\sin\omega}{\cos\omega} = \frac{e^{i\omega} - e^{-i\omega}}{i(e^{i\omega} + e^{-i\omega})}$$

$$\therefore iz(e^{i\omega} + e^{-i\omega}) = e^{i\omega} - e^{-i\omega} \quad * e^{i\omega}$$

$$iz(e^{2i\omega} + 1) = e^{2i\omega} - 1$$

$$(iz - 1)\, e^{2i\omega} = -1 - iz$$

$$e^{2i\omega} = \frac{1+iz}{1-iz} * \frac{i}{i} = \frac{i-z}{i+z}$$

$$\therefore 2i\omega = \ln\frac{i-z}{i+z} \Rightarrow \omega = \frac{1}{2i} \ln\frac{i-z}{i+z}$$

$$\omega = -\frac{i}{2} \ln\left(\frac{i+z}{i-z}\right) = \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right)^{-1}$$

$$\therefore \tan^{-1} z = \frac{i}{2} \ln\frac{i-z}{i+z}$$

• Show that $\sinh^{-1} z = \ln(z + \sqrt{z^2+1})$

Sol.

$$\omega = \sinh^{-1} z \Rightarrow z = \sinh\omega \Rightarrow \frac{e^{\omega} - e^{-\omega}}{2} = z$$

$$\therefore e^{\omega} - e^{-\omega} = 2z \quad * e^{\omega}$$

$$e^{2\omega} - 2z e^{\omega} - 1 = 0$$

$$\therefore e^{\omega} = \frac{\cancel{2}z \pm \sqrt{4z^2+4}}{\cancel{2}} = z \pm \sqrt{z^2+1}$$

refused

$$\therefore \omega = \ln(z + \sqrt{z^2+1})$$

$$\therefore \sinh^{-1} z = \ln(z + \sqrt{z^2+1})$$

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• Show that $\cosh^{-1} z = \ln(z + \sqrt{z^2-1})$

$$w = \cosh^{-1} z \Rightarrow \cosh w = z \xRightarrow{\text{Sol.}} \frac{e^{w}+e^{-w}}{2} = z$$

$$\therefore e^{w} + e^{-w} = 2z \quad (*e^{w})$$

$$e^{2w} - 2ze^{w} + 1 = 0$$

$$e^{w} = \frac{\cancel{2}z \pm \sqrt{4z^2-4}}{\cancel{2}}$$

$$e^{w} = z \pm \sqrt{z^2-1}$$

$\hookrightarrow$ refused

$$\therefore e^{w} = z + \sqrt{z^2-1} \Rightarrow w = \ln(z + \sqrt{z^2-1})$$

$$\therefore \cosh^{-1} z = \ln(z + \sqrt{z^2-1})$$

• Show that $\tanh^{-1} z = \frac{1}{2} \ln \frac{1+z}{1-z}$

$$w = \tanh^{-1} z \Rightarrow z = \tanh w \overset{\text{Sol.}}{=} \frac{\sinh w}{\cosh w} = \frac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$$

$$\therefore (e^{w} + e^{-w}) z = (e^{w} - e^{-w}) \quad (*e^{w})$$

$$\therefore e^{2w}(z-1) = -1 - z$$

$$\therefore e^{2w} = \frac{1+z}{1-z}$$

$$\therefore 2w = \ln \frac{1+z}{1-z}$$

$$\therefore w = \frac{1}{2} \ln \frac{1+z}{1-z} \Rightarrow \tanh^{-1} z = \frac{1}{2} \ln \frac{1+z}{1-z}$$

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Integration

• Integral of a complex-valued function of a real variable

هذا أبسط أنواع التكامل حيث تكون الدالة المركبة "$f(z)=u+iv$" تعتمد على متغير واحد فقط بدلا من متغيرين ويتم ذلك عندما تكون كلا من الدالتين "u,v" حقيقية (real) بمتغير واحد "t".

$$\boxed{\int_a^b f(t) = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt}$$

• Properities

1- $\operatorname{Re}\int_a^b f(t)\,dt = \int_a^b \operatorname{Re}(f(t))\,dt = \int_a^b u(t)\,dt$

2- $\operatorname{Im}\int_a^b f(t)\,dt = \int_a^b \operatorname{Im}(f(t))\,dt = \int_a^b v(t)\,dt$

3- $\int_a^b [\alpha f(t) + \beta g(t)]\,dt = \alpha\int_a^b f(t)\,dt + \beta\int_a^b g(t)\,dt$

4- $\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$

5- $\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt$

• ex Find $\int_0^{\pi/3} e^{it}\,dt$

Sol.

$$I = \int_0^{\pi/3}(\cos t + i\sin t)\,dt$$

$$= \int_0^{\pi/3}\cos t\,dt + i\int_0^{\pi/3}\sin t\,dt$$

$$= \sin t\Big|_0^{\frac{\pi}{3}} - i\cos t\Big|_0^{\frac{\pi}{3}} = \frac{\sqrt{2}}{2} + \frac{i}{2}$$

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- Line integration

يتم التكامل على منحنى "C" في الـ Complex plane ولذلك سوف نقوم بمعرفة أنواع الـ Curves أولاً.

- Simple Curve

$z(t_1) \neq z(t_2)$

$\forall \; t_1 \neq t_2$

(منحنى لا يقطع نفسه)

- Closed Curve

$z(a) = z(b)$

بداية المنحنى هي نهاية المنحنى

- Smooth Curve

$x'(t)$ & $y'(t) \neq 0$

at a time

المشتقات لا تنعدم في نفس الوقت

- Contour → Curve consisting of a finite number of smooth Curves joined end to end.

- Integration on Curve (Line integral)

$$\boxed{\int_c f(z)\,dz = \int_c (u\,dx - v\,dy) + i \int_c (u\,dy + v\,dx)}$$

- ex. Evaluate $\int_c \bar{z}\,dz$ where C is given by $x = 3t$ and $y = t^2$

$-1 \leq t \leq 4$

Sol.

$z(t) = 3t + it^2 \Rightarrow \bar{z}(t) = 3t - it^2 \Rightarrow dz = (3 + i2t)\,dt$

$$\therefore I = \int_{-1}^{4} (3t + it^2)(3 + 2it)\,dt$$

$$= \int_{-1}^{4} (9t + 9it^2 - 2t^3)\,dt$$

$$= \frac{9}{2}t^2 + 3it^3 - \frac{1}{2}t^4 \Big|_{-1}^{4}$$

$$\boxed{I = -52 + 189i}$$

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2- Evaluate the integral

$$\oint_C (8x^2 - iy)\,dz \quad \text{on } y = 5x \qquad 0 \leq x \leq 2$$

Sol.

$$z(\,) = x + i5x \Rightarrow dz = (1+5i)\,dx$$

$$\therefore I = \int_0^2 (8x^2 - 5ix)(1+5i)\,dx$$

$$= (1+5i)\left[\frac{8}{3}x^3 - \frac{5}{2}ix^2\right]_0^2$$

$$I = (1+5i)\left(\frac{64}{3} - \frac{5}{2}i\right)$$

$$\therefore I = \frac{203 + 625i}{6}$$

• Properities of definite integrals

1- $\int_C (\alpha f(z) + \beta g(z))\,dz = \alpha \int_C f(z)\,dz + \beta \int_C g(z)\,dz$

2- if "C" consists of a contour C_1 joining a to c and a cont. C_2 joining c to b then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

3- $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

the same set of Points taken in inverse order

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• Bounding theorem (ML-inequality)

If $f(z)$ is continuous on a smooth curve "C" and if

$$|f(z)| \leq M$$

for all z on C, then

$$\left| \int_C f(z)\,dz \right| \leq ML$$

where: "M" is the upper bound of $|f(z)|$ and L is the length of "C".

• Proof

$$\left| \int_C f(z)\,dz \right| \leq \int_C |f(z)\,dz|$$

$$\leq \int_C |f(z)|\,|dz|$$

$$\leq \int_C M\,|dz| \leq M \int_C |dz|$$

$$\leq ML \;\#$$

• Note

طول المسار $\boxed{L = \int_C \sqrt{(x')^2+(y')^2}\,dt}$

• ex Find an upper for the absolute value of $\oint_{|z|=2} \frac{e^z}{z^2+1}\,dz$

Sol.

$$L = 2\pi r = 2\pi(2) = 4\pi$$

$$\left|\frac{e^z}{z^2+1}\right| = \frac{|e^z|}{|z^2+1|} \leq \frac{e^4}{|z|^2-|1|} \leq \frac{e^4}{4-1} = \frac{e^4}{3}$$

$$\boxed{\therefore |I| \leq \frac{4\pi e^4}{3}}$$

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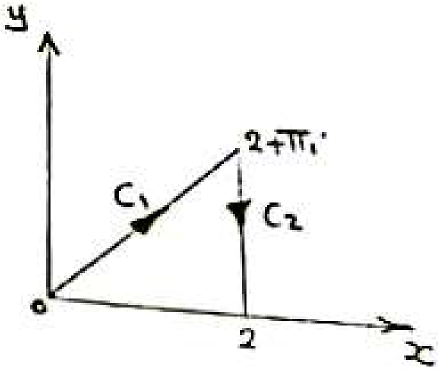
ex. Find $\int_C f(z)\,dz$ where $f(z) = e^{iz}$ and C: two straight lines from $0, 2+\pi i, 2$ respectively



Sol.

$$I = \int_C e^{iz}\,dz = \int_{C_1} e^{iz}\,dz + \int_{C_2} e^{iz}\,dz$$

$$= \int_0^{2+\pi i} e^{iz}\,dz + \int_{2+\pi i}^{2} e^{iz}\,dz$$

$$= \frac{e^{iz}}{i}\Big|_0^{2+\pi i} + \frac{e^{iz}}{i}\Big|_{2+\pi i}^{2}$$

$$= \frac{e^{iz}}{i}\Big|_0^{2} = -i(e^{2i}-1)$$

---

ex. Show that

$$\left|\int_C (x^2+iy^2)\,dz\right| \leq \pi$$

where C the right half circle $|z|=1$ and $Re(z) \geq 0$

Sol.

$$\left|\int_C (x^2+iy^2)\,dz\right| \leq \int_C |x^2+iy^2|\cdot|dz|$$

$$\leq \int_C (|x^2|+|y^2|)\cdot|dz|$$

$$\leq \int_C |z|^2\cdot|dz|$$

$$\leq \int_C (1)\cdot|dz|$$

$$\leq \int_C |dz| \leq \pi(1) \leq \boxed{\pi}$$

طول المسار

المسار هو نصف دائرة ⇐ طوله $= \pi r$

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• Independence of the path

let z_0 and z_1 be points in a domain D. A Contour integral

$$\int_C f(z)\,dz$$

is said to be independent of the path if its value is the same for all contours C in D with an initial point z_0 and a terminal point z_1.

• Theorem: (Analyticity implies path independence)

If $f(z)$ is an analytic function in a simply connected domain D, then

$$\int_C f(z)\,dz$$

is independent of the path C.

• ex. Show that

$$\int_C e^{-2z}\,dz$$

is independent of the path "C" joining $1-\pi i$ and $2+\pi i$ and determine its value

Sol.

$$f(z) = e^{-2z} = e^{-2x} \cdot e^{-2iy} = e^{-2x}(\cos 2y - i\sin 2y)$$

$$\therefore\ u = e^{-2x}\cos 2y \qquad v = -e^{-2x}\sin 2y$$

$$u_x = -2e^{-2x}\cos 2y \qquad v_x = 2e^{-2x}\sin 2y$$

$$u_y = -2e^{-2x}\sin 2y \qquad v_y = -2e^{-2x}\cos 2y$$

C-R is satisfied & u, v, u_x, u_y, v_x, v_y are conti.

then $f(z)$ is analytic $\Rightarrow I$ is independent on C

ex. Evaluate

$$\int_C \frac{1}{z^2}\,dz \quad \text{along the Following two paths}$$

I 1- the Straight line Segment joining 1 and 2+i

2- the horizontal line From 1 to 2, then the vertical line From 2 to 2+i

Sol.

1- $$\int_1^{2+i} \frac{1}{z^2}\,dz = \frac{-1}{z}\Big|_1^{2+i} = \frac{-1}{2+i} + 1 = \frac{3+i}{5}$$

2- $f(z)$ is analytic on the path and the path starts From 1 to 2+i $\Rightarrow \int_{C_2} f(z)\,dz$ is independent on Path c.

$$\Rightarrow \int_{C_2} \frac{1}{z^2}\,dz = \int_{C_1} \frac{1}{z^2}\,dz = \frac{3+i}{5}$$

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ex. Find $\int_C (z+2)\,e^{iz}\,dz$ along the parabola "C" defined by $\pi y^2 = x^2$ From (0,0) to $(\pi,1)$

Sol.

$f(z) = (z+2)\,e^{iz}$ is entire $\Rightarrow$ It's independent on "C"

$$\therefore I = \int_0^{\pi+i} (z+2)\,e^{iz}\,dz$$

$u = z+2 \qquad dv = e^{iz}\,dz$

$du = dz \qquad v = -ie^{iz}$

$$I = (z+2)\frac{e^{iz}}{i}\Big|_0^{\pi+i} + i\int_0^{\pi+i} e^{iz}\,dz$$

$$= \frac{(z+2)}{i}e^{iz}\Big|_0^{\pi+i} + e^{iz}\Big|_0^{\pi+i} = \frac{(2+\pi+i)}{i}e^{(i\pi-1)} + 2i + (e^{i\pi-1} - 1)$$

$\ldots \bar{z})\,dz$

• **Theorem: Green's theorem in the plane**

let $P(x,y)$ and $Q(x,y)$ be Continuous and have Continuous partial derivatives in a region R and on its boundary "C". Green's theorem States that

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x}\right) dx\,dy$$

• **Theorem: Cauchy - Goursat theorem**

Given a simple closed Contour C and let $f(z)$ is analytic on and inside "C", then

$$\oint_C f(z)\,dz = 0$$

• **Proof**

Since $f(z) = u + iv$ is analytic and has a Continuous derivative $f'(z) = u_x + i v_x$ it Follows that the partial derivatives $u_x = v_y$ and $u_y = -v_x$ are Continuous inside and on C. Thus Green's theorem Can be applied and we have

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx + i\,dy)$$

$$= \oint_C u\,dx - v\,dy + i\oint v\,dx + u\,dy$$

$$= \iint_R \underbrace{\left(-\frac{\partial u}{\partial x} - \cancel{\frac{\partial u}{\partial y}}\right)}_{0} dx\,dy + i\iint_R \underbrace{\left(\frac{\partial u}{\partial x} - \cancel{\frac{\partial u}{\partial y}}\right)}_{0} dx\,dy$$

$$\therefore \boxed{\oint_C f(z)\,dz = 0}$$

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Evaluate

1- $\oint_{|z|=2} \frac{e^z}{(z^2-9)} dz$

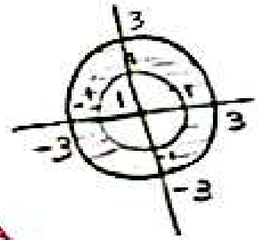
Sol.

$f(z)$ isn't analytic at $z = \pm 3$ and these pts outside circle $|z| = 2$ $\Rightarrow$ $f(z)$ is analytic on and inside C

$\therefore \; I = 0$

2- $\oint_C \frac{e^z}{z(z^2-16)} dz$ where C is the boundary of the annulus between the circles of radius 1 and radius 3

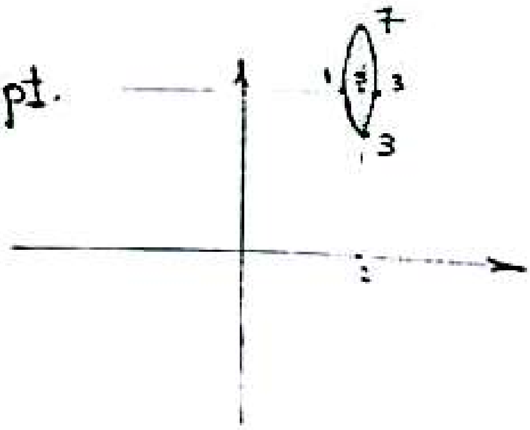
Sol.



$f(z) \Rightarrow$ isn't analytic at $z = 0$ and $z = \pm 4$ and these pts outside region $\Rightarrow$ $f(z)$ is analytic on and inside C $\Rightarrow$ $I = 0$

3- $\oint_C \frac{1}{z^2} dz$ where $C: (x-2)^2 + \frac{(y-5)^2}{4} = 1$

Sol.



$f(z)$ isn't analytic at $z = 0$ and that pt. outside region $\Rightarrow$ $f(z)$ is analytic on and inside C

$\therefore \; I = 0$

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4- $\oint_{|z|=a} \left(|z| - e^{z}\sin z^{2} + \bar{z}\right) dz$

Sol.

$$I = \oint_C |z|\,dz - \oint_C e^{z}\sin z^{2} + \oint_C \bar{z}\,dz$$

$$= \underbrace{\oint_C a\,dz}_{0} - \underbrace{\oint_C e^{z}\sin z^{2}}_{0} + \oint_C \bar{z}\,dz$$

$$\oint_{|z|=a} \bar{z}\,dz = \oint_{|z|=a} a^{2} e^{-i\theta}\cdot i e^{i\theta}\,d\theta = a^{2} i\int_0^{2\pi} d\theta = 2\pi a^{2} i$$

$$z = a e^{i\theta} \Rightarrow \bar{z} = a e^{-i\theta} \Rightarrow dz = a i e^{i\theta}\,d\theta$$

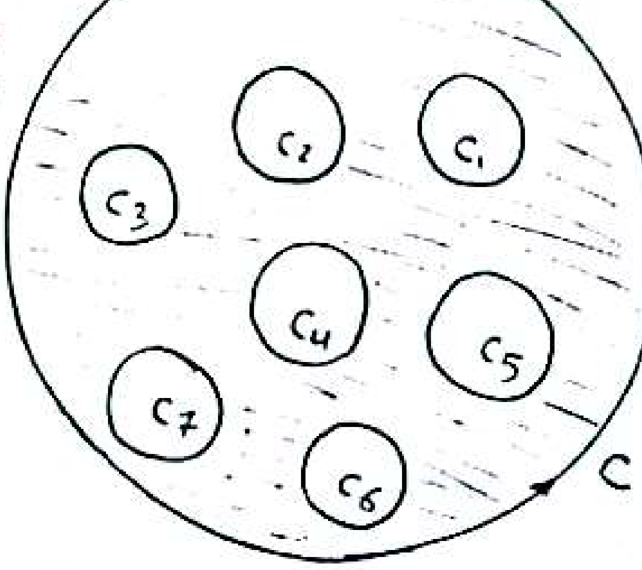
$$\therefore \boxed{I = 2\pi a^{2} i}$$

• Theorem: Cauchy-Goursat for multiply connected domains.

Suppose $C_1, C_2, C_3, \ldots, C_n$ are simple closed curves with a positive orientation such that $C_1, C_2, C_3, \ldots, C_n$ are interior but the regions interior to each C_k, $k = 1, 2, \ldots n$ have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, \ldots, n$ then

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$$

$f(z)$ analytic on C_k and C between them



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• Theorem. Cauchy's Integral Formula

let "f" be analytic in a simply connected domain D. and let "C" be a simple closed contour lying entirely within D. If z_0 is any point within C, then

$$1.\quad \oint_C \frac{f(z)}{(z-z_0)}\,dz = 2\pi i f(z_0)$$

$$2.\quad \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

• Proof

let D be a simple connected domain, C a simple closed contour in D and z_0 an interior point of C. Let Γ be a circle centered at z_0 with radius small enough that it interior to C. We can write

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_\Gamma \frac{f(z)}{z-a}\,dz$$

$$\therefore \oint_\Gamma \frac{f(z)}{z-a}\,dz = \oint_\Gamma \frac{f(z_0)-f(z_0)+f(z)}{(z-a)}\,dz$$

$$= f(z_0)\oint_\Gamma \frac{1}{z-a} + \oint_\Gamma \frac{f(z)-f(z_0)}{z-a}\,dz$$

$$= 2\pi i f(z_0) + \oint_\Gamma \frac{f(z)-f(z_0)}{z-a}\,dz$$

Since $f(z)$ is continuous at z_0 for any arbitrarily small $\varepsilon>0$ there exists a $\delta>0$ $\ni$

$$|f(z)-f(z_0)|<\varepsilon \quad \text{whenever } |z-z_0|<\delta$$

if we choose the circle Γ to be

$$|z-z_0| = \frac{\delta}{2} < \delta$$

then by ML-inequality, the absolute value of the integral on

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right Side.

$$\left| \oint_{\Gamma} \frac{f(z)-f(z_0)}{z-z_0} dz \right| \leq \oint_{\Gamma} \left| \frac{f(z)-f(z_0)}{z-z_0} \right| |dz|$$

$$= \frac{\epsilon}{\delta/2} \cdot 2\pi \cdot \frac{\delta}{2} = 2\pi\epsilon$$

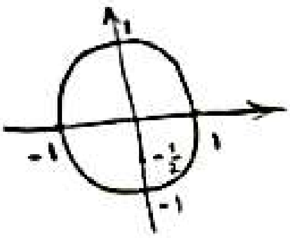
Since ϵ can be chosen arbitrary small, it follows that the last integral has the value zero and we have

$$\oint \frac{f(z)}{z-z_0} dz = 2\pi i \, f(z_0)$$

• Evaluate

1- $\oint_C \frac{z^2 e^z}{2z+i}$ where C is the unit circle $|z|=1$ traversed in the clockwise direction.

Sol.



$$z_0 = -\frac{i}{2}$$

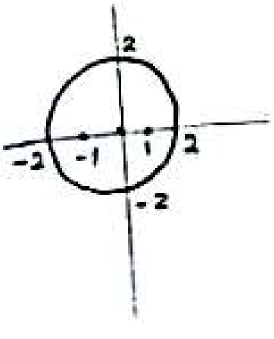
$$\therefore I = -\oint_C \frac{(z^2 e^z)/2}{z-(-\frac{i}{2})} dz$$

$$\therefore f(z) = \frac{z^2 e^z}{2} \implies I = -2\pi i \left(\frac{z_0^2 e^{z_0}}{2} \right)$$

$$I = \frac{\pi i}{4} e^{-i/2}$$

2- $\oint_{|z|=2} \frac{e^z}{z^2-1} dz$

Sol.



$$I = \oint_{|z|=2} \frac{e^z}{(z-1)(z+1)} dz$$

$$= \oint_{|z|=2} \frac{e^z/(z+1)}{z-1} dz + \oint_{|z|=2} \frac{e^z/(z-1)}{(z+1)} dz$$

$$= 2\pi i \left[\frac{e^z}{z+1} \right]_{z=1} + 2\pi i \left[\frac{e^z}{z-1} \right]_{z=-1}$$

$$= \pi i \left[e - e^{-1} \right]$$

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• Compute the integral

$$\int_C \frac{z+1}{z^4+4z^3}\,dz$$

where C is the unit Circle $|z|=1$

Sol.

$f(z)$ isn't analytic at $z=0$, $z=-4 \Rightarrow z_0=0$ since $z_0=-4$ outside region

$$\therefore I = \oint_C \frac{(z+1)}{z^3(z+4)}\,dz = \oint_C \frac{(z+1)/(z+4)}{z^3}\,dz$$

$$\therefore f(z) = \frac{(z+1)}{(z+4)} \quad , \quad n=2$$

$$\therefore I = \frac{2\pi i}{2!} f''(0) = \frac{2\pi i}{2!}\left(\frac{-6}{(0+4)^3}\right)$$

$$\therefore I = -\frac{3\pi}{32} i$$

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